COHOMOLOGIES OF CERTAIN ORBIFOLDS

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Abstract. We study the Bott-Chern cohomology of complex orbifolds obtained as quotient of a compact complex manifold by a finite group of biholomorphisms.

Introduction

In order to investigate cohomological aspects of compact complex non-Kähler manifolds, and in particular with the aim to get results allowing to construct new examples of non-Kähler manifolds, we study the cohomology of complex orbifolds.

Namely, an *orbifold* (or *V-manifold*, as introduced by I. Satake, [21]) is a singular complex space whose singularities are locally isomorphic to quotient singularities $\mathbb{C}^n / G$, for finite subgroups $G \subset \mathrm{GL}(n;\mathbb{C})$, where n is the complex dimension: in other words, local geometry of orbifolds reduces to local G-invariant geometry. A special case is provided by orbifolds of global-quotient type, namely, by orbifolds $\tilde{X} = X / G$, where X is a complex manifold and G is a finite group of biholomorphisms of X; such orbifolds have been studied, among others, by D. D. Joyce in constructing examples of compact manifolds with special holonomy, see [13, 12, 14, 15]. As proven by I. Satake, and W. L. Baily, from the cohomological point of view, one can adapt both the sheaf-theoretic and the analytic tools for the study of the de Rham and Dolbeault cohomology of complex orbifolds, [21, 4, 5].

In particular, an useful tool in studying the cohomological properties of non-Kähler manifolds is provided by the *Bott-Chern cohomology*, that is, the bi-graded algebra

$$H^{\bullet,\bullet}_{BC}(X) := \frac{\ker\partial \cap \ker\overline{\partial}}{\operatorname{im}\partial\overline{\partial}} .$$

While for compact Kähler manifolds X one has that the Bott-Chern cohomology is naturally isomorphic to the Dolbeault cohomology, [8, Lemma 5.15, Remark 5.16, 5.21, Lemma 5.11], in general, for compact non-Kähler manifolds X, the natural maps $H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet,\bullet}_{\overline{\partial}}(X)$ and $H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet}_{dR}(X;\mathbb{C})$ induced by the identity are neither injective nor surjective. One says that a compact complex manifold *satisfies the $\partial\overline{\partial}$-Lemma* if every ∂-closed $\overline{\partial}$-closed d-exact form is $\partial\overline{\partial}$-exact, that is, if the natural map $H^{\bullet,\bullet}_{BC}(X) \to H^{\bullet}_{dR}(X;\mathbb{C})$ is injective; compact Kähler manifolds provide the main examples of complex manifolds satisfying the $\partial\overline{\partial}$-Lemma, [8, Lemma 5.11], other than motivations for their study.

In this note, we study the *Bott-Chern cohomology* of compact complex orbifolds $\tilde{X} = X / G$ of global-quotient type, (where X is a compact complex manifold and G is a finite group of biholomorphisms of X,) that is, the bi-graded $\mathbb{C}$-algebra

$$H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) := \frac{\ker\partial \cap \ker\overline{\partial}}{\operatorname{im}\partial\overline{\partial}}$$

where $\partial\colon \wedge^{\bullet,\bullet}\tilde{X} \to \wedge^{\bullet+1,\bullet}\tilde{X}$ and $\overline{\partial}\colon \wedge^{\bullet,\bullet}\tilde{X} \to \wedge^{\bullet,\bullet+1}\tilde{X}$, and $\wedge^{\bullet,\bullet}\tilde{X}$ is the bi-graded $\mathbb{C}$-vector space of *differential forms* on $\tilde{X}$, that is, of G-invariant differential forms on X. We prove the following result, see Theorem 2.1.

Theorem. *Let $\tilde{X} = X / G$ be a compact complex orbifold of complex dimension n, where X is a compact complex manifold and G is a finite group of biholomorphisms of X. For any $p, q \in \mathbb{N}$, there is a canonical isomorphism*

$$H^{p,q}_{BC}\left(\tilde{X}\right) \simeq \frac{\ker\left(\partial\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p+1,q}\tilde{X}\right) \cap \ker\left(\overline{\partial}\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p,q+1}\tilde{X}\right)}{\operatorname{im}\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right)} ,$$

where $\mathcal{D}^{p,q}\tilde{X}$ denotes the space of currents of bi-degree (p,q) on $\tilde{X}$, that is, the space of G-invariant currents of bi-degree (p,q) on X.

Furthermore, given a Hermitian metric on $\tilde{X}$ (that is, a G-invariant Hermitian metric on X), there are canonical isomorphisms

$$H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) \simeq \ker\tilde{\Delta}_{BC} \qquad \text{and} \qquad H^{\bullet,\bullet}_{A}\left(\tilde{X}\right) \simeq \ker\tilde{\Delta}_{A} ,$$

where $\tilde{\Delta}_{BC}$ and $\tilde{\Delta}_{A}$ are the 4^{th} order self-adjoint elliptic differential operators

$$\tilde{\Delta}_{BC} := \left(\partial\overline{\partial}\right)\left(\partial\overline{\partial}\right)^* + \left(\partial\overline{\partial}\right)^*\left(\partial\overline{\partial}\right) + \left(\overline{\partial}^*\partial\right)\left(\overline{\partial}^*\partial\right)^* + \left(\overline{\partial}^*\partial\right)^*\left(\overline{\partial}^*\partial\right) + \overline{\partial}^*\overline{\partial} + \partial^*\partial \in \operatorname{End}\left(\wedge^{\bullet,\bullet}\tilde{X}\right)$$

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and

$$\tilde{\Delta}_A := \partial\partial^* + \overline{\partial}\overline{\partial}^* + \left(\partial\overline{\partial}\right)^* \left(\partial\overline{\partial}\right) + \left(\partial\overline{\partial}\right) \left(\partial\overline{\partial}\right)^* + \left(\overline{\partial}\partial^*\right)^* \left(\overline{\partial}\partial^*\right) + \left(\overline{\partial}\partial^*\right) \left(\overline{\partial}\partial^*\right)^* \in \mathrm{End}\left(\wedge^{\bullet,\bullet}\tilde{X}\right) .$$

In particular, the Hodge-$*$-operator induces an isomorphism

$$H^{\bullet_1,\bullet_2}_{BC}\left(\tilde{X}\right) \simeq H^{n-\bullet_2,n-\bullet_1}_{A}\left(\tilde{X}\right) .$$

As regards the $\partial\overline{\partial}$-Lemma for complex orbifolds, by adapting a result by R. O. Wells in [23], we get the following result, see Corollary 3.2.

Corollary. *Let $\tilde{Y}$ and $\tilde{X}$ be compact complex orbifolds of the same complex dimension, and let $\epsilon\colon \tilde{Y} \to \tilde{X}$ be a proper surjective morphism of complex orbifolds. If $\tilde{Y}$ satisfies the $\partial\overline{\partial}$-Lemma, then also $\tilde{X}$ satisfies the $\partial\overline{\partial}$-Lemma.*

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1. Preliminaries on orbifolds

The notion of orbifold has been introduced by I. Satake in [21], with the name of *V-manifold*, and has been studied, among many others, by W. L. Baily, [4, 5].

In this section, we start by recalling the main definitions and some classical results concerning complex orbifolds and their cohomology, referring to [16, 15, 21, 4, 5].

A *complex orbifold of complex dimension n* is a singular complex space whose singularities are locally isomorphic to quotient singularities $\mathbb{C}^n / G$, for finite subgroups $G \subset \mathrm{GL}(n;\mathbb{C})$, [21, Definition 2].

By definition, an object (e.g., a *differential form*, a *Riemannian metric*, a *Hermitian metric*) *on a complex orbifold* $\tilde{X}$ is defined locally at $x \in \tilde{X}$ as a G_x-invariant object on $\mathbb{C}^n$, where $G_x \subseteq \mathrm{GL}(n;\mathbb{C})$ is such that $\tilde{X}$ is locally isomorphic to $\mathbb{C}^n / G_x$ at x.

Given $\tilde{X}$ and $\tilde{Y}$ complex orbifolds, a *morphism $f\colon \tilde{Y} \to \tilde{X}$ of complex orbifolds* is a morphism of complex spaces given, locally at $y \in \tilde{Y}$, by a map $\mathbb{C}^m / H_y \to \mathbb{C}^n / G_{f(y)}$, where $\tilde{Y}$ is locally isomorphic to $\mathbb{C}^m / H_y$ at y and $\tilde{X}$ is locally isomorphic to $\mathbb{C}^n / G_{f(y)}$ at $f(y)$.

In particular, one gets a differential complex $\left(\wedge^{\bullet}\tilde{X},\, \mathrm{d}\right)$, and a double complex $\left(\wedge^{\bullet,\bullet}\tilde{X},\, \partial,\, \overline{\partial}\right)$. Define the de Rham, Dolbeault, Bott-Chern, and Aeppli cohomology groups of $\tilde{X}$ respectively as

$$H^{\bullet}_{dR}\left(\tilde{X};\mathbb{C}\right) := \frac{\ker \mathrm{d}}{\mathrm{im}\, \mathrm{d}} , \qquad H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{X}\right) := \frac{\ker \overline{\partial}}{\mathrm{im}\, \overline{\partial}} ,$$

$$H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) := \frac{\ker \partial \cap \ker \overline{\partial}}{\mathrm{im}\, \partial\overline{\partial}} , \qquad H^{\bullet,\bullet}_{A}\left(\tilde{X}\right) := \frac{\ker \partial\overline{\partial}}{\mathrm{im}\, \partial + \mathrm{im}\, \overline{\partial}} .$$

The structure of double complex of $\left(\wedge^{\bullet,\bullet}\tilde{X},\, \partial,\, \overline{\partial}\right)$ induces naturally a spectral sequence $\left\{\left(E_r^{\bullet,\bullet},\, \mathrm{d}_r\right)\right\}_{r\in\mathbb{N}}$, called *Hodge and Frölicher spectral sequence of* $\tilde{X}$, such that $E_1^{\bullet,\bullet} \simeq H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{X}\right)$ (see, e.g., [19, §2.4]). Hence, one has the *Frölicher inequality*, see [11, Theorem 2],

$$\sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}\left(\tilde{X}\right) \geq \dim_{\mathbb{C}} H^{k}_{dR}\left(\tilde{X};\mathbb{C}\right) ,$$

for any $k \in \mathbb{N}$.

Given a Riemannian metric on a complex orbifold $\tilde{X}$ of complex dimension n, one can consider the $\mathbb{R}$-linear Hodge-$*$-operator $*_g\colon \wedge^{\bullet}\tilde{X} \to \wedge^{2n-\bullet}\tilde{X}$, and hence the 2^{nd} order self-adjoint elliptic differential operator $\Delta := [\mathrm{d}, \mathrm{d}^*] := \mathrm{d}\,\mathrm{d}^* + \mathrm{d}^*\,\mathrm{d} \in \mathrm{End}\left(\wedge^{\bullet}\tilde{X}\right)$.

Analogously, given a Hermitian metric on a complex orbifold $\tilde{X}$ of complex dimension n, one can consider the $\mathbb{C}$-linear Hodge-$*$-operator $*_g\colon \wedge^{\bullet_1,\bullet_2}\tilde{X} \to \wedge^{n-\bullet_2,n-\bullet_1}\tilde{X}$, and hence the 2^{nd} order self-adjoint elliptic differential operator $\overline{\square} := \left[\overline{\partial}, \overline{\partial}^*\right] := \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial} \in \mathrm{End}\left(\wedge^{\bullet,\bullet}\tilde{X}\right)$. Furthermore, in [17, Proposition 5], and [22, §2], the following 4^{th} order self-adjoint elliptic differential operators are defined:

$$\tilde{\Delta}_{BC} := \left(\partial\overline{\partial}\right)\left(\partial\overline{\partial}\right)^* + \left(\partial\overline{\partial}\right)^*\left(\partial\overline{\partial}\right) + \left(\overline{\partial}^*\partial\right)\left(\overline{\partial}^*\partial\right)^* + \left(\overline{\partial}^*\partial\right)^*\left(\overline{\partial}^*\partial\right) + \overline{\partial}^*\overline{\partial} + \partial^*\partial \in \mathrm{End}\left(\wedge^{\bullet,\bullet}\tilde{X}\right)$$

and

$$\tilde{\Delta}_A := \partial\partial^* + \overline{\partial}\overline{\partial}^* + \left(\partial\overline{\partial}\right)^* \left(\partial\overline{\partial}\right) + \left(\partial\overline{\partial}\right) \left(\partial\overline{\partial}\right)^* + \left(\overline{\partial}\partial^*\right)^* \left(\overline{\partial}\partial^*\right) + \left(\overline{\partial}\partial^*\right) \left(\overline{\partial}\partial^*\right)^* \in \mathrm{End}\left(\wedge^{\bullet,\bullet}\tilde{X}\right) .$$

As a matter of notation, given a compact complex orbifold $\tilde{X}$ of complex dimension n, denote the constant sheaf with coefficients in $\mathbb{R}$ over $\tilde{X}$ by $\underline{\mathbb{R}}_{\tilde{X}}$, the sheaf of germs of smooth functions over $\tilde{X}$ by $\mathcal{C}^{\infty}_{\tilde{X}}$, the sheaf of germs of (p,q)-forms (for $p,q \in \mathbb{N}$) over $\tilde{X}$ by $\mathcal{A}^{p,q}_{\tilde{X}}$, the sheaf of germs of k-forms (for $k \in \mathbb{N}$) over $\tilde{X}$ by $\mathcal{A}^{k}_{\tilde{X}}$, the sheaf of germs

of bidimension-(p,q)-currents (for $p,q\in\mathbb{N}$) over $\tilde{X}$ by $\mathcal{D}_{\tilde{X}\,p,q} := \mathcal{D}^{n-p,n-q}_{\tilde{X}}$, the sheaf of germs of dimension-k-currents (for $k\in\mathbb{N}$) over $\tilde{X}$ by $\mathcal{D}_{\tilde{X}\,k} := \mathcal{D}^{2n-k}_{\tilde{X}}$, and the sheaf of holomorphic p-forms (for $p\in\mathbb{N}$) over $\tilde{X}$ by $\Omega^p_{\tilde{X}}$.

The following result, concerning the de Rham cohomology of a compact complex orbifold, has been proven by I. Satake, [21], and by W. L. Baily, [4].

Theorem 1.1 ([21, Theorem 1], [4, Theorem H]). *Let $\tilde{X}$ be a compact complex orbifold of complex dimension n. There is a canonical isomorphism*

$$H^{\bullet}_{dR}\left(\tilde{X};\mathbb{R}\right) \;\simeq\; \check{H}^{\bullet}\left(\tilde{X};\underline{\mathbb{R}}_{\tilde{X}}\right) .$$

Furthermore, given a Riemannian metric on $\tilde{X}$, there is a canonical isomorphism

$$H^{\bullet}_{dR}\left(\tilde{X};\mathbb{R}\right) \;\simeq\; \ker\Delta .$$

In particular, the Hodge-$*$-operator induces an isomorphism

$$H^{\bullet}_{dR}\left(\tilde{X};\mathbb{R}\right) \;\simeq\; H^{2n-\bullet}_{dR}\left(\tilde{X};\mathbb{R}\right) .$$

The isomorphism $H^{\bullet}_{dR}\left(\tilde{X};\mathbb{R}\right)\simeq\ker\Delta$ can be seen as a consequence of a more general decomposition theorem on compact orbifolds, [4, Theorem D], which holds for 2^{nd} order self-adjoint elliptic differential operators. In particular, as regards the Dolbeault cohomology, the following result by W. L. Baily, [5, 4], holds.

Theorem 1.2 ([5, page 807], [4, Theorem K]). *Let $\tilde{X}$ be a compact complex orbifold of complex dimension n. There is a canonical isomorphism*

$$H^{\bullet_1,\bullet_2}_{\overline{\partial}}\left(\tilde{X}\right) \;\simeq\; \check{H}^{\bullet_2}\left(\tilde{X};\Omega^{\bullet_1}_{\tilde{X}}\right) .$$

Furthermore, given a Hermitian metric on X, there is a canonical isomorphism

$$H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{X}\right) \;\simeq\; \ker\overline{\Box} .$$

In particular, the Hodge-$*$-operator induces an isomorphism

$$H^{\bullet_1,\bullet_2}_{\overline{\partial}}\left(\tilde{X}\right) \;\simeq\; H^{n-\bullet_1,n-\bullet_2}_{\overline{\partial}}\left(\tilde{X}\right) .$$

2. Bott-Chern cohomology of complex orbifolds of global-quotient type

Compact complex orbifolds of the type $\tilde{X}=X/G$, where X is a compact complex manifold and G is a finite group of biholomorphisms of X, constitute one of the simplest examples of singular manifolds: more precisely, in this section, we study the Bott-Chern cohomology for such orbifolds, proving that it can be defined using either currents or forms, or also by computing the G-invariant $\tilde{\Delta}_{BC}$-harmonic forms on X, Theorem 2.1.

Consider

$$\tilde{X} \;=\; X/G ,$$

where X is a compact complex manifold and G is a finite group of biholomorphisms of X: by the Bochner linearization theorem, [6, Theorem 1], see also [20, Theorem 1.7.2], $\tilde{X}$ turns out to be an orbifold as in I. Satake's definition.

Such orbifolds of global-quotient type have been considered and studied by D. D. Joyce in constructing examples of compact 7-dimensional manifolds with holonomy G_2, [13] and [15, Chapters 11-12], and examples of compact 8-dimensional manifolds with holonomy $\mathrm{Spin}(7)$, [12, 14] and [15, Chapters 13-14]. See also [10, 7] for the use of orbifolds of global-quotient type to construct a compact 8-dimensional simply-connected non-formal symplectic manifold (which do not satisfy, respectively satisfy, the Hard Lefschetz condition), answering to a question by I. K. Babenko and I. A. Taĭmanov, [3, Problem].

Since G is a finite group of biholomorphisms, the singular set of $\tilde{X}$ is

$$\mathrm{Sing}\left(\tilde{X}\right) \;=\; \left\{xG\in X/G \;:\; x\in X \text{ and } g\cdot x=x \text{ for some } g\in G\setminus\{\mathrm{id}_X\}\right\} .$$

We provide the following result, concerning Bott-Chern and Aeppli cohomologies of compact complex orbifolds of global-quotient type.

Theorem 2.1. *Let $\tilde{X}=X/G$ be a compact complex orbifold of complex dimension n, where X is a compact complex manifold and G is a finite group of biholomorphisms of X. For any $p,q\in\mathbb{N}$, there is a canonical isomorphism*

$$(1)\qquad H^{p,q}_{BC}\left(\tilde{X}\right) \;\simeq\; \frac{\ker\left(\partial\colon\mathcal{D}^{p,q}\tilde{X}\to\mathcal{D}^{p+1,q}\tilde{X}\right)\cap\ker\left(\overline{\partial}\colon\mathcal{D}^{p,q}\tilde{X}\to\mathcal{D}^{p,q+1}\tilde{X}\right)}{\mathrm{im}\left(\partial\overline{\partial}\colon\mathcal{D}^{p-1,q-1}\tilde{X}\to\mathcal{D}^{p,q}\tilde{X}\right)} .$$

Furthermore, given a Hermitian metric on $\tilde{X}$, there are canonical isomorphisms

$$H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) \;\simeq\; \ker\tilde{\Delta}_{BC} \qquad\text{and}\qquad H^{\bullet,\bullet}_{A}\left(\tilde{X}\right) \;\simeq\; \ker\tilde{\Delta}_{A} .$$

In particular, the Hodge-$*$-operator induces an isomorphism

$$H^{\bullet_1,\bullet_2}_{BC}\left(\tilde{X}\right) \;\simeq\; H^{n-\bullet_2,n-\bullet_1}_{A}\left(\tilde{X}\right) .$$

Proof. We use the same argument as in the proof of [1, Theorem 3.7] to show that, since the de Rham cohomology and the Dolbeault cohomology of $\tilde{X}$ can be computed using either differential forms or currents, the same holds true for the Bott-Chern and the Aeppli cohomologies.

Indeed, note that, for any $p, q \in \mathbb{N}$, one has the exact sequence

$$0 \to \frac{\mathrm{im}\left(\mathrm{d}\colon \left(\mathcal{D}^{p+q-1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\mathcal{D}^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)\cap\mathcal{D}^{p,q}\tilde{X}}{\mathrm{im}\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X}\to\mathcal{D}^{p,q}\tilde{X}\right)}$$

$$\to \frac{\ker\left(\mathrm{d}\colon \mathcal{D}^{p,q}\tilde{X}\to\mathcal{D}^{p+1,q+1}\tilde{X}\right)}{\mathrm{im}\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X}\to\mathcal{D}^{p,q}\tilde{X}\right)} \to \frac{\ker\left(\mathrm{d}\colon \left(\mathcal{D}^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\mathcal{D}^{p+q+1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)}{\mathrm{im}\left(\mathrm{d}\colon \left(\mathcal{D}^{p+q-1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\mathcal{D}^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)} ,$$

where the maps are induced by the identity. By [21, Theorem 1], one has

$$\frac{\ker\left(\mathrm{d}\colon \left(\mathcal{D}^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\mathcal{D}^{p+q+1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)}{\mathrm{im}\left(\mathrm{d}\colon \left(\mathcal{D}^{p+q-1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\mathcal{D}^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)} \;\simeq\; \frac{\ker\left(\mathrm{d}\colon \left(\wedge^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\wedge^{p+q+1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)}{\mathrm{im}\left(\mathrm{d}\colon \left(\wedge^{p+q-1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\wedge^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)} ,$$

therefore it suffices to prove that the space

$$\frac{\mathrm{im}\left(\mathrm{d}\colon \left(\mathcal{D}^{p+q-1}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\to\left(\mathcal{D}^{p+q}\tilde{X}\otimes_{\mathbb{R}}\mathbb{C}\right)\right)\cap\mathcal{D}^{p,q}\tilde{X}}{\mathrm{im}\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X}\to\mathcal{D}^{p,q}\tilde{X}\right)}$$

can be computed using just differential forms on $\tilde{X}$.

Firstly, we note that, since, by [5, page 807],

$$\frac{\ker\left(\overline{\partial}\colon \mathcal{D}^{\bullet,\bullet}\tilde{X}\to\mathcal{D}^{\bullet,\bullet+1}\tilde{X}\right)}{\mathrm{im}\left(\overline{\partial}\colon \mathcal{D}^{\bullet,\bullet-1}\tilde{X}\to\mathcal{D}^{\bullet,\bullet}\tilde{X}\right)} \;\simeq\; \frac{\ker\left(\overline{\partial}\colon \wedge^{\bullet,\bullet}\tilde{X}\to\wedge^{\bullet,\bullet+1}\tilde{X}\right)}{\mathrm{im}\left(\overline{\partial}\colon \wedge^{\bullet,\bullet-1}\tilde{X}\to\wedge^{\bullet,\bullet}\tilde{X}\right)} ,$$

one has that, if $\psi \in \wedge^{r,s}\tilde{X}$ is a $\overline{\partial}$-closed differential form, then every solution $\phi \in \mathcal{D}^{r,s-1}$ of $\overline{\partial}\phi = \psi$ is a differential form up to $\overline{\partial}$-exact terms. Indeed, since $[\psi] = 0$ in $\frac{\ker\overline{\partial}\cap\mathcal{D}^{r,s}\tilde{X}}{\mathrm{im}\,\overline{\partial}}$ and hence in $\frac{\ker\overline{\partial}\cap\wedge^{r,s}\tilde{X}}{\mathrm{im}\,\overline{\partial}}$, there is a differential form $\alpha \in \wedge^{r,s-1}\tilde{X}$ such that $\psi = \overline{\partial}\alpha$. Hence, $\phi - \alpha \in \mathcal{D}^{r,s-1}\tilde{X}$ defines a class in $\frac{\ker\overline{\partial}\cap\mathcal{D}^{r,s-1}\tilde{X}}{\mathrm{im}\,\overline{\partial}} \simeq \frac{\ker\overline{\partial}\cap\wedge^{r,s-1}\tilde{X}}{\mathrm{im}\,\overline{\partial}}$, and hence $\phi - \alpha$ is a differential form up to a $\overline{\partial}$-exact form, and so ϕ is.

By conjugation, if $\psi \in \wedge^{r,s}\tilde{X}$ is a ∂-closed differential form, then every solution $\phi \in \mathcal{D}^{r-1,s}$ of $\partial\phi = \psi$ is a differential form up to ∂-exact terms.

Now, let

$$\omega^{p,q} \;=\; \mathrm{d}\,\eta \quad \mathrm{mod}\ \mathrm{im}\,\partial\overline{\partial} \;\in\; \frac{\mathrm{im}\,\mathrm{d}\cap\mathcal{D}^{p,q}X}{\mathrm{im}\,\partial\overline{\partial}} .$$

Decomposing $\eta =: \sum_{p,q}\eta^{p,q}$ in pure-type components, where $\eta^{p,q} \in \mathcal{D}^{p,q}\tilde{X}$, the previous equality is equivalent to the system

$$\begin{cases} & \partial\eta^{p+q-1,0} & = & 0 & \mathrm{mod}\ \mathrm{im}\,\partial\overline{\partial} & \\ \overline{\partial}\eta^{p+q-\ell,\ell-1} & +\ \partial\eta^{p+q-\ell-1,\ell} & = & 0 & \mathrm{mod}\ \mathrm{im}\,\partial\overline{\partial} & \text{for} \quad \ell\in\{1,\ldots,q-1\} \\ \overline{\partial}\eta^{p,q-1} & +\ \partial\eta^{p-1,q} & = & \omega^{p,q} & \mathrm{mod}\ \mathrm{im}\,\partial\overline{\partial} & \\ \overline{\partial}\eta^{\ell,p+q-\ell-1} & +\ \partial\eta^{\ell-1,p+q-\ell} & = & 0 & \mathrm{mod}\ \mathrm{im}\,\partial\overline{\partial} & \text{for} \quad \ell\in\{1,\ldots,p-1\} \\ \overline{\partial}\eta^{0,p+q-1} & & = & 0 & \mathrm{mod}\ \mathrm{im}\,\partial\overline{\partial} & \end{cases} .$$

By the above argument, we may suppose that, for $\ell \in \{0,\ldots,p-1\}$, the currents $\eta^{\ell,p+q-\ell-1}$ are differential form: indeed, they are differential form up to $\overline{\partial}$-exact terms, but $\overline{\partial}$-exact terms give no contribution in the system, which is modulo $\mathrm{im}\,\partial\overline{\partial}$. Analogously, we may suppose that, for $\ell \in \{0,\ldots,q-1\}$, the currents $\eta^{p+q-\ell-1,\ell}$ are differential form. Then we may suppose that $\omega^{p,q} = \overline{\partial}\eta^{p,q-1} + \partial\eta^{p-1,q}$ is a differential form. Hence (1) is proven.

Now, we prove that, fixed a G-invariant Hermitian metric on $\tilde{X}$, the Bott-Chern cohomology of $\tilde{X}$ is isomorphic to the space of $\tilde{\Delta}_{BC}$-harmonic G-invariant forms on X. Indeed, since the elements of G commute with ∂, $\overline{\partial}$, ∂^*, and $\overline{\partial}^*$, and hence with $\tilde{\Delta}_{BC}$, the following decomposition, [22, Théorème 2.2],

$$\wedge^{\bullet,\bullet}X \;=\; \ker\tilde{\Delta}_{BC}\oplus\partial\overline{\partial}\wedge^{\bullet-1,\bullet-1}X\oplus\left(\partial^*\wedge^{\bullet+1,\bullet}X+\overline{\partial}^*\wedge^{\bullet,\bullet+1}X\right)$$

induces a decomposition

$$\wedge^{\bullet,\bullet}\tilde{X} \;=\; \ker\tilde{\Delta}_{BC}\oplus\partial\overline{\partial}\wedge^{\bullet-1,\bullet-1}\tilde{X}\oplus\left(\partial^*\wedge^{\bullet+1,\bullet}\tilde{X}+\overline{\partial}^*\wedge^{\bullet,\bullet+1}\tilde{X}\right) ;$$

more precisely, let $\alpha \in \wedge^{\bullet,\bullet}\tilde{X}$, that is, α is a G-invariant form on X; if α has a decomposition $\alpha = h_\alpha + \partial\overline{\partial}\beta + \left(\partial^*\gamma + \overline{\partial}^*\eta\right)$ with $h_\alpha, \beta, \gamma, \eta \in \wedge^{\bullet,\bullet}X$ such that $\tilde{\Delta}_{BC}h_\alpha = 0$, then one has

$$\alpha \;=\; \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\alpha \;=\; \left(\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*h_\alpha\right) + \partial\overline{\partial}\left(\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\beta\right) + \left(\partial^*\left(\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\gamma\right) + \overline{\partial}^*\left(\eta\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\right)\right) ,$$

where $\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*h_\alpha$, $\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\beta$, $\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\gamma$, $\eta\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^* \in \wedge^{\bullet,\bullet}\tilde{X}$ and

$$\tilde{\Delta}_{BC}\left(\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*h_\alpha\right) \;=\; \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\left(\tilde{\Delta}_{BC}h_\alpha\right) \;=\; 0 .$$

As regards the Aeppli cohomology, one has the decomposition, [22, §2.c],

$$\wedge^{\bullet,\bullet}X \;=\; \ker\tilde{\Delta}_A \oplus \left(\partial\wedge^{\bullet-1,\bullet}X + \overline{\partial}\wedge^{\bullet,\bullet-1}X\right) \oplus \left(\partial\overline{\partial}\right)^*\wedge^{\bullet+1,\bullet+1}X ,$$

and hence the decomposition

$$\wedge^{\bullet,\bullet}\tilde{X} \;=\; \ker\tilde{\Delta}_A \oplus \left(\partial\wedge^{\bullet-1,\bullet}\tilde{X} + \overline{\partial}\wedge^{\bullet,\bullet-1}\tilde{X}\right) \oplus \left(\partial\overline{\partial}\right)^*\wedge^{\bullet+1,\bullet+1}\tilde{X} ,$$

from which one gets the isomorphism $H^{\bullet,\bullet}_A\left(\tilde{X}\right) \simeq \ker\tilde{\Delta}_A$.

Finally, note that the Hodge-$*$-operator $*\colon \wedge^{\bullet_1,\bullet_2}\tilde{X} \to \wedge^{n-\bullet_2,n-\bullet_1}\tilde{X}$ sends $\tilde{\Delta}_{BC}$-harmonic forms to $\tilde{\Delta}_A$-harmonic forms, and hence it induces an isomorphism

$$*\colon H^{\bullet_1,\bullet_2}_{BC}\left(\tilde{X}\right) \stackrel{\simeq}{\to} H^{n-\bullet_2,n-\bullet_1}_A\left(\tilde{X}\right) ,$$

concluding the proof. □

Remark 2.2. We note that another proof of the isomorphism

$$H^{p,q}_{BC}\left(\tilde{X}\right) \;\simeq\; \frac{\ker\left(\partial\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p+1,q}\tilde{X}\right) \cap \ker\left(\overline{\partial}\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p,q+1}\tilde{X}\right)}{\operatorname{im}\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right)} ,$$

and a proof of the isomorphism

$$H^{p,q}_{A}\left(\tilde{X}\right) \;\simeq\; \frac{\ker\left(\partial\overline{\partial}\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p+1,q+1}\tilde{X}\right)}{\operatorname{im}\left(\partial\colon \mathcal{D}^{p-1,q}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right) + \operatorname{im}\left(\overline{\partial}\colon \mathcal{D}^{p,q-1}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right)}$$

follow from the sheaf-theoretic interpretation of the Bott-Chern and Aeppli cohomologies, developed by J.-P. Demailly, [9, §V I.12.1] and M. Schweitzer, [22, §4], see also [18, §3.2].

We recall that, for any $p, q \in \mathbb{N}$, the complex $\left(\mathcal{L}^\bullet_{\tilde{X}\,p,q}, \mathrm{d}_{\mathcal{L}^\bullet_{\tilde{X}\,p,q}}\right)$ of sheaves is defined as

$$\left(\mathcal{L}^\bullet_{\tilde{X}\,p,q}, \mathrm{d}_{\mathcal{L}^\bullet_{\tilde{X}\,p,q}}\right) \;:\; \mathcal{A}^{0,0}_{\tilde{X}} \stackrel{\mathrm{pr}\circ\mathrm{d}}{\to} \bigoplus_{\substack{r+s=1\\ r<p,\, s<q}} \mathcal{A}^{r,s}_{\tilde{X}} \to \cdots \stackrel{\mathrm{pr}\circ\mathrm{d}}{\to} \bigoplus_{\substack{r+s=p+q-2\\ r<p,\, s<q}} \mathcal{A}^{r,s}_{\tilde{X}} \stackrel{\partial\overline{\partial}}{\to} \bigoplus_{\substack{r+s=p+q\\ r\geq p,\, s\geq q}} \mathcal{A}^{r,s}_{\tilde{X}} \stackrel{\mathrm{d}}{\to} \bigoplus_{\substack{r+s=p+q\\ r\geq p,\, s\geq q}} \mathcal{A}^{r,s}_{\tilde{X}} \to \cdots ,$$

and the complex $\left(\mathcal{M}^\bullet_{\tilde{X}\,p,q}, \mathrm{d}_{\mathcal{M}^\bullet_{\tilde{X}\,p,q}}\right)$ of sheaves is defined as

$$\left(\mathcal{M}^\bullet_{\tilde{X}\,p,q}, \mathrm{d}_{\mathcal{M}^\bullet_{\tilde{X}\,p,q}}\right) \;:\; \mathcal{D}^{0,0}_{\tilde{X}} \stackrel{\mathrm{pr}\circ\mathrm{d}}{\to} \bigoplus_{\substack{r+s=1\\ r<p,\, s<q}} \mathcal{D}^{r,s}_{\tilde{X}} \to \cdots \stackrel{\mathrm{pr}\circ\mathrm{d}}{\to} \bigoplus_{\substack{r+s=p+q-2\\ r<p,\, s<q}} \mathcal{D}^{r,s}_{\tilde{X}} \stackrel{\partial\overline{\partial}}{\to} \bigoplus_{\substack{r+s=p+q\\ r\geq p,\, s\geq q}} \mathcal{D}^{r,s}_{\tilde{X}} \stackrel{\mathrm{d}}{\to} \bigoplus_{\substack{r+s=p+q\\ r\geq p,\, s\geq q}} \mathcal{D}^{r,s}_{\tilde{X}} \to \cdots ,$$

where pr denotes the projection onto the appropriate space.

Take ϕ a germ of a d-closed k-form on $\tilde{X}$, with $k \in \mathbb{N}\setminus\{0\}$, that is, a germ of a G-invariant k-form on X; by the Poincaré lemma, see, e.g., [9, I.1.22], there exists ψ a germ of a $(k-1)$-form on X such that $\phi = \mathrm{d}\,\psi$; since ϕ is G-invariant, one has

$$\phi \;=\; \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\phi \;=\; \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\left(\mathrm{d}\,\psi\right) \;=\; \mathrm{d}\left(\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\psi\right) ,$$

that is, taking the germ of the G-invariant $(k-1)$-form

$$\tilde{\psi} \;:=\; \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\psi$$

on X, one gets a germ of a $(k-1)$-form on $\tilde{X}$ such that $\phi = \mathrm{d}\,\tilde{\psi}$. As regards the case $k=0$, one has straightforwardly that every (G-invariant) d-closed function on X is locally constant. The same argument applies for the sheaves of currents, by using the Poincaré lemma for currents, see, e.g., [9, Theorem I.2.24].

Analogously, take ϕ a germ of a $\overline{\partial}$-closed (p,q)-form (respectively, bidimension-(p,q)-current) on $\tilde{X}$, with $q \in \mathbb{N}\setminus\{0\}$, that is, a germ of a G-invariant (p,q)-form (respectively, bidimension-(p,q)-current) on X; by the Dolbeault and Grothendieck lemma, see, e.g., [9, I.3.29], there exists ψ a germ of a $(p,q-1)$-form (respectively, bidimension-$(p,q-1)$-current) on X such that $\phi = \overline{\partial}\psi$; since ϕ is G-invariant, one has

$$\phi \;=\; \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\phi \;=\; \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\left(\overline{\partial}\psi\right) \;=\; \overline{\partial}\left(\frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\psi\right) \;,$$

that is, taking the germ of the G-invariant $(p,q-1)$-form (respectively, bidimension-$(p,q-1)$-current)

$$\tilde{\psi} \;:=\; \frac{1}{\operatorname{ord} G}\sum_{g\in G} g^*\psi$$

on X, one gets a germ of a $(p,q-1)$-form (respectively, bidimension-$(p,q-1)$-current) on $\tilde{X}$ such that $\phi = \overline{\partial}\tilde{\psi}$. As regards the case $q=0$, one has that every (G-invariant) $\overline{\partial}$-closed bidimension-$(p,0)$-current on X is locally a holomorphic p-form, see, e.g., [9, I.3.29].

By the Poincaré lemma and the Dolbeault and Grothendieck lemma, one gets M. Schweitzer's lemma [22, Lemme 4.1], which can be extended also to the context of orbifolds by using the same trick; this allows to prove that the map

$$\left(\mathcal{L}^\bullet_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^\bullet_{\tilde{X}\,p,q}}\right) \to \left(\mathcal{M}^\bullet_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{M}^\bullet_{\tilde{X}\,p,q}}\right)$$

of complexes of sheaves is a quasi-isomorphism, and hence, see, e.g., [9, §IV.12.6], for every $\ell \in \mathbb{N}$,

$$\mathbb{H}^\ell\left(\tilde{X};\, \left(\mathcal{L}^\bullet_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^\bullet_{\tilde{X}\,p,q}}\right)\right) \;\simeq\; \mathbb{H}^\ell\left(\tilde{X};\, \left(\mathcal{M}^\bullet_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^\bullet_{\tilde{X}\,p,q}}\right)\right) \;.$$

Since, for every $k \in \mathbb{N}$, the sheaves $\mathcal{L}^k_{\tilde{X}\,p,q}$ and $\mathcal{M}^k_{\tilde{X}\,p,q}$ are fine (indeed, they are sheaves of $\left(\mathcal{C}^\infty_{\tilde{X}} \otimes_{\mathbb{R}} \mathbb{C}\right)$-modules over a paracompact space), one has, see, e.g., [9, IV.4.19, (IV.12.9)],

$$\mathbb{H}^{p+q-1}\left(\tilde{X};\, \left(\mathcal{L}^\bullet_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^\bullet_{\tilde{X}\,p,q}}\right)\right) \;\simeq\; \frac{\ker\left(\partial\colon\; \wedge^{p,q}\tilde{X} \to \wedge^{p+1,q}\tilde{X}\right) \cap \ker\left(\overline{\partial}\colon\; \wedge^{p,q}\tilde{X} \to \wedge^{p,q+1}\tilde{X}\right)}{\operatorname{im}\left(\partial\overline{\partial}\colon\; \wedge^{p-1,q-1}\tilde{X} \to \wedge^{p,q}\tilde{X}\right)}$$

and

$$\mathbb{H}^{p+q-1}\left(\tilde{X};\, \left(\mathcal{M}^\bullet_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^\bullet_{\tilde{X}\,p,q}}\right)\right) \;\simeq\; \frac{\ker\left(\partial\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p+1,q}\tilde{X}\right) \cap \ker\left(\overline{\partial}\colon \mathcal{D}^{p,q}\tilde{X} \to \mathcal{D}^{p,q+1}\tilde{X}\right)}{\operatorname{im}\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right)} \;,$$

and

$$\mathbb{H}^{p+q-2}\left(\tilde{X};\, \left(\mathcal{L}^\bullet_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^\bullet_{\tilde{X}\,p,q}}\right)\right) \;\simeq\; \frac{\ker\left(\partial\overline{\partial}\colon\; \wedge^{p-1,q-1}\tilde{X} \to \wedge^{p,q}\tilde{X}\right)}{\operatorname{im}\left(\partial\colon\; \wedge^{p-2,q-1}\tilde{X} \to \wedge^{p-1,q-1}\tilde{X}\right) + \operatorname{im}\left(\overline{\partial}\colon\; \wedge^{p-1,q-2}\tilde{X} \to \wedge^{p-1,q-1}\tilde{X}\right)}$$

and

$$\mathbb{H}^{p+q-2}\left(\tilde{X};\, \left(\mathcal{M}^\bullet_{\tilde{X}\,p,q},\, \mathrm{d}_{\mathcal{L}^\bullet_{\tilde{X}\,p,q}}\right)\right) \;\simeq\; \frac{\ker\left(\partial\overline{\partial}\colon \mathcal{D}^{p-1,q-1}\tilde{X} \to \mathcal{D}^{p,q}\tilde{X}\right)}{\operatorname{im}\left(\partial\colon \mathcal{D}^{p-2,q-1}\tilde{X} \to \mathcal{D}^{p-1,q-1}\tilde{X}\right) + \operatorname{im}\left(\overline{\partial}\colon \mathcal{D}^{p-1,q-2}\tilde{X} \to \mathcal{D}^{p-1,q-1}\tilde{X}\right)} \;,$$

proving the stated isomorphisms.

3. Complex orbifolds satisfying the $\partial\overline{\partial}$-Lemma

We recall that a bounded double complex $\left(K^{\bullet,\bullet},\, \mathrm{d}',\, \mathrm{d}''\right)$ of vector spaces, whose associated simple complex is $(K^\bullet,\, \mathrm{d})$ with $\mathrm{d} := \mathrm{d}' + \mathrm{d}''$, is said *to satisfy the* $\mathrm{d}'\,\mathrm{d}''$*-Lemma*, [8], if

$$\ker \mathrm{d}' \cap \ker \mathrm{d}'' \cap \operatorname{im} \mathrm{d} = \operatorname{im} \mathrm{d}'\,\mathrm{d}'' \;;$$

other equivalent conditions are provided in [8, Lemma 5.15].

An orbifold $\tilde{X}$ is said *to satisfy the* $\partial\overline{\partial}$*-Lemma* if the double complex $\left(\wedge^{\bullet,\bullet}\tilde{X},\, \partial,\, \overline{\partial}\right)$ satisfies the $\partial\overline{\partial}$-Lemma, that is, if every ∂-closed $\overline{\partial}$-closed d-exact form is $\partial\overline{\partial}$-exact, namely, in other words, if the natural map $H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) \to H^\bullet_{dR}\left(\tilde{X};\mathbb{C}\right)$ induced by the identity is injective.

Characterizations of compact complex manifolds satisfying the $\partial\overline{\partial}$-Lemma in terms of their cohomological properties have been provided by P. Deligne, Ph. Griffiths, J. Morgan and D. Sullivan in [8, Proposition 5.17, 5.21], and by the author and A. Tomassini in [2, Theorem B]. As a corollary of their characterization, P. Deligne, Ph. Griffiths, J. Morgan and D. Sullivan proved that, given X and Y compact complex manifolds of the same dimension and $f\colon X \to Y$ a holomorphic birational map, if X satisfies the $\partial\overline{\partial}$-Lemma, then also Y satisfies the $\partial\overline{\partial}$-Lemma, [8, Theorem 5.22].

In this section, we extend [8, Theorem 5.22] to the case of orbifolds, by straightforwardly adapting a result by R. O. Wells, [23, Theorem 3.1], to the orbifold case.

Theorem 3.1 (see [23, Theorem 3.1])**.** *Let $\tilde{Y}$ and $\tilde{X}$ be compact complex orbifolds of the same complex dimension, and let $\epsilon\colon \tilde{Y} \to \tilde{X}$ be a proper surjective morphism of complex orbifolds. Then the map $\epsilon\colon \tilde{Y} \to \tilde{X}$ induces injective maps*

$$\epsilon^*_{dR}\colon H^\bullet_{dR}\left(\tilde{X};\mathbb{R}\right) \to H^\bullet_{dR}\left(\tilde{Y};\mathbb{R}\right) \;, \qquad \epsilon^*_{\overline{\partial}}\colon H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{X}\right) \to H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{Y}\right) \;, \qquad \text{and} \qquad \epsilon^*_{BC}\colon H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) \to H^{\bullet,\bullet}_{BC}\left(\tilde{Y}\right) \;.$$

Proof. We follow closely the proof of [23, Theorem 3.1] and adapt it to the orbifold case.

`Step 1` – *Notations.* The morphism $\epsilon\colon \tilde{Y} \to \tilde{X}$ of complex orbifolds induces morphisms

$$\epsilon^*\colon \wedge^\bullet \tilde{X} \to \wedge^\bullet \tilde{Y} \qquad \text{and} \qquad \epsilon^*\colon \wedge^{\bullet,\bullet} \tilde{X} \to \wedge^{\bullet,\bullet}\tilde{Y}$$

of $\mathbb{R}$-vector spaces and $\mathbb{C}$-vector spaces, and hence, by duality,

$$\epsilon_*\colon \mathcal{D}_\bullet \tilde{Y} \to \mathcal{D}_\bullet \tilde{X} \qquad \text{and} \qquad \epsilon_*\colon \mathcal{D}_{\bullet,\bullet}\tilde{Y} \to \mathcal{D}_{\bullet,\bullet}\tilde{X} \;.$$

Moreover, recall that, for $X \in \left\{\tilde{X},\, \tilde{Y}\right\}$, there are natural inclusions

$$T_\cdot\colon \wedge^\bullet X \to \mathcal{D}^\bullet X := \mathcal{D}_{2n-\bullet} X \qquad \text{and} \qquad T_\cdot\colon \wedge^{\bullet,\bullet} X \to \mathcal{D}^{\bullet,\bullet} X := \mathcal{D}_{n-\bullet,n-\bullet} X \;,$$

where n is the complex dimension of X.

Both ϵ^* and ϵ_* commute with d, ∂ and $\overline{\partial}$, and hence they induce morphisms of complexes

$$\epsilon^*_{dR}\colon \left(\wedge^\bullet \tilde{X},\, \mathrm{d}\right) \to \left(\wedge^\bullet \tilde{Y},\, \mathrm{d}\right) \qquad \text{and} \qquad \epsilon^{dR}_*\colon \left(\mathcal{D}^\bullet \tilde{Y},\, \mathrm{d}\right) \to \left(\mathcal{D}^\bullet \tilde{X},\, \mathrm{d}\right) \;,$$

and, for any $p \in \mathbb{N}$,

$$\epsilon^*_{\overline{\partial}}\colon \left(\wedge^{p,\bullet}\tilde{X},\, \overline{\partial}\right) \to \left(\wedge^{p,\bullet}\tilde{Y},\, \overline{\partial}\right) \qquad \text{and} \qquad \epsilon^{\overline{\partial}}_*\colon \left(\mathcal{D}^{p,\bullet}\tilde{Y},\, \overline{\partial}\right) \to \left(\mathcal{D}^{p,\bullet}\tilde{X},\, \overline{\partial}\right) \;,$$

and, for any $p, q \in \mathbb{N}$,

$$\epsilon^*_{BC}\colon \left(\wedge^{p-1,q-1}\tilde{X} \stackrel{\partial\overline{\partial}}{\to} \wedge^{p,q}\tilde{X} \stackrel{\partial+\overline{\partial}}{\to} \wedge^{p+1,q}\tilde{X} \oplus \wedge^{p,q+1}\tilde{X}\right) \to \left(\wedge^{p-1,q-1}\tilde{Y} \stackrel{\partial\overline{\partial}}{\to} \wedge^{p,q}\tilde{Y} \stackrel{\partial+\overline{\partial}}{\to} \wedge^{p+1,q}\tilde{Y} \oplus \wedge^{p,q+1}\tilde{Y}\right)$$

and

$$\epsilon^{BC}_*\colon \left(\mathcal{D}^{p-1,q-1}\tilde{Y} \stackrel{\partial\overline{\partial}}{\to} \mathcal{D}^{p,q}\tilde{Y} \stackrel{\partial+\overline{\partial}}{\to} \mathcal{D}^{p+1,q}\tilde{Y} \oplus \wedge^{p,q+1}\tilde{Y}\right) \to \left(\mathcal{D}^{p-1,q-1}\tilde{X} \stackrel{\partial\overline{\partial}}{\to} \mathcal{D}^{p,q}\tilde{X} \stackrel{\partial+\overline{\partial}}{\to} \mathcal{D}^{p+1,q}\tilde{X} \oplus \mathcal{D}^{p,q+1}\tilde{X}\right) \;;$$

hence, they induce morphisms between the corresponding cohomologies:

$$\epsilon^*_{dR}\colon H^\bullet_{dR}\left(\tilde{X};\mathbb{R}\right) \to H^\bullet_{dR}\left(\tilde{Y};\mathbb{R}\right) \;, \quad \epsilon^*_{\overline{\partial}}\colon H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{X}\right) \to H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{Y}\right) \;, \quad \text{and} \quad \epsilon^*_{BC}\colon H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) \to H^{\bullet,\bullet}_{BC}\left(\tilde{Y}\right) \;.$$

Recall that $T_\cdot$ commutes with d, ∂ and $\overline{\partial}$, and hence it induces, for $X \in \left\{\tilde{X},\, \tilde{Y}\right\}$, morphisms

$$T_\cdot\colon \left(\wedge^\bullet X,\, \mathrm{d}\right) \to \left(\mathcal{D}^\bullet X,\, \mathrm{d}\right) \;,$$

and, for any $p \in \mathbb{N}$,

$$T_\cdot\colon \left(\wedge^{p,\bullet} X,\, \overline{\partial}\right) \to \left(\mathcal{D}^{p,\bullet} X,\, \overline{\partial}\right) \;,$$

and, for any $p, q \in \mathbb{N}$,

$$T_\cdot\colon \left(\wedge^{p-1,q-1} X \stackrel{\partial\overline{\partial}}{\to} \wedge^{p,q} X \stackrel{\partial+\overline{\partial}}{\to} \wedge^{p+1,q} X \oplus \wedge^{p,q+1} X\right) \to \left(\mathcal{D}^{p-1,q-1} X \stackrel{\partial\overline{\partial}}{\to} \mathcal{D}^{p,q} X \stackrel{\partial+\overline{\partial}}{\to} \mathcal{D}^{p+1,q} X \oplus \wedge^{p,q+1} X\right) \;;$$

by [21, Theorem 1], by [5, page 807], and by Theorem 2.1, these maps are in fact quasi-isomorphisms.

`Step 3` – *It holds $\epsilon_*\, T_\cdot\, \epsilon^* = \mu \cdot T_\cdot$ for some $\mu \in \mathbb{N} \setminus \{0\}$.* Indeed, consider the diagrams

$$\begin{tikzcd} \wedge^\bullet \tilde{Y} \arrow[r, "T_\cdot"] & \mathcal{D}^\bullet \tilde{Y} \arrow[d, "\epsilon_*"] \\ \wedge^\bullet \tilde{X} \arrow[u, "\epsilon^*"] \arrow[r, "T_\cdot"'] & \mathcal{D}^\bullet \tilde{X} \end{tikzcd} \;, \qquad \text{respectively} \qquad \begin{tikzcd} \wedge^{\bullet,\bullet} \tilde{Y} \arrow[r, "T_\cdot"] & \mathcal{D}^{\bullet,\bullet} \tilde{Y} \arrow[d, "\epsilon_*"] \\ \wedge^{\bullet,\bullet} \tilde{X} \arrow[u, "\epsilon^*"] \arrow[r, "T_\cdot"'] & \mathcal{D}^{\bullet,\bullet} \tilde{X} \end{tikzcd} \;.$$

One has that there exists a proper analytic subset $S_{\tilde{Y}}$ of $\tilde{Y} \setminus \mathrm{Sing}\left(\tilde{Y}\right)$ such that $S_{\tilde{Y}}$ has measure zero in $\tilde{Y}$ and

$$\epsilon\lfloor_{\tilde{Y}\setminus\left(\mathrm{Sing}\left(\tilde{Y}\right)\cup S_{\tilde{Y}}\right)}\colon \tilde{Y} \setminus \left(\mathrm{Sing}\left(\tilde{Y}\right) \cup S_{\tilde{Y}}\right) \to \tilde{X} \setminus \left(\mathrm{Sing}\left(\tilde{X}\right) \cup \epsilon\left(S_{\tilde{Y}}\right)\right)$$

is a finitely-sheeted covering mapping of sheeting number $\mu \in \mathbb{N} \setminus \{0\}$. Let $\mathcal{U} :=: \{U_\alpha\}_{\alpha\in\mathcal{A}}$ be an open covering of $\tilde{X} \setminus \left(\mathrm{Sing}\left(\tilde{X}\right) \cup \epsilon\left(S_{\tilde{Y}}\right)\right)$, and let $\{\rho_\alpha\}_{\alpha\in\mathcal{A}}$ be an associated partition of unity. For every $\varphi, \psi \in \wedge^{\bullet,\bullet}\tilde{X}$, one has that

$$\begin{aligned}
\langle \epsilon_* T.\, \epsilon^*\varphi,\, \psi\rangle &= \langle T.\, \epsilon^*\varphi,\, \epsilon^*\psi\rangle \;=\; \int_{\tilde{Y}} \epsilon^*\varphi \wedge \epsilon^*\psi \;=\; \int_{\tilde{Y}} \epsilon^*\left(\varphi\wedge\psi\right) \;=\; \int_{\tilde{Y}-\left(\mathrm{Sing}\left(\tilde{Y}\right)\cup S_{\tilde{Y}}\right)} \epsilon^*\left(\varphi\wedge\psi\right) \\
&= \sum_{\alpha\in\mathcal{A}} \int_{\pi^{-1}(U_\alpha)} \epsilon^*\left(\rho_\alpha\left(\varphi\wedge\psi\right)\right) \;=\; \sum_{\alpha\in\mathcal{A}} \sum_{\sharp\{U\in\mathcal{U}\,:\,\pi^{-1}(U)=\pi^{-1}(U_\alpha)\}} \int_{U_\alpha} \rho_\alpha\left(\varphi\wedge\psi\right) \\
&= \mu\cdot\int_{\tilde{X}-\left(\mathrm{Sing}\left(\tilde{X}\right)\cup\epsilon\left(S_{\tilde{Y}}\right)\right)} \varphi\wedge\psi \;=\; \mu\cdot\int_{\tilde{X}} \varphi\wedge\psi \;=\; \langle \mu\, T.\varphi,\, \psi\rangle \;,
\end{aligned}$$

and hence one gets that

$$\epsilon_*\, T.\, \epsilon^* \;=\; \mu\cdot T. \;.$$

`Step 4` – *Conclusion.* Hence one has the diagrams

$$\begin{array}{ccc}
\dfrac{\ker\left(\mathrm{d}\colon \wedge^{\bullet}\tilde{X}\to\wedge^{\bullet+1}\tilde{X}\right)}{\mathrm{im}\left(\mathrm{d}\colon \wedge^{\bullet-1}\tilde{X}\to\wedge^{\bullet}\tilde{X}\right)} & \xrightarrow[\simeq]{T.} & \dfrac{\ker\left(\mathrm{d}\colon \mathcal{D}^{\bullet}\tilde{X}\to\mathcal{D}^{\bullet+1}\tilde{X}\right)}{\mathrm{im}\left(\mathrm{d}\colon \mathcal{D}^{\bullet}\tilde{X}\to\mathcal{D}^{\bullet+1}\tilde{X}\right)} \\
\Big\uparrow{\scriptstyle \epsilon^*_{dR}} & & \Big\downarrow{\scriptstyle \epsilon_*^{dR}} \\
\dfrac{\ker\left(\mathrm{d}\colon \wedge^{\bullet}\tilde{Y}\to\wedge^{\bullet+1}\tilde{Y}\right)}{\mathrm{im}\left(\mathrm{d}\colon \wedge^{\bullet-1}\tilde{Y}\to\wedge^{\bullet}\tilde{Y}\right)} & \xrightarrow[T.]{\simeq} & \dfrac{\ker\left(\mathrm{d}\colon \mathcal{D}^{\bullet}\tilde{Y}\to\mathcal{D}^{\bullet+1}\tilde{Y}\right)}{\mathrm{im}\left(\mathrm{d}\colon \mathcal{D}^{\bullet}\tilde{Y}\to\mathcal{D}^{\bullet+1}\tilde{Y}\right)}
\end{array} \;,$$

such that

$$\epsilon_*^{dR}\, T.\, \epsilon^*_{dR} \;=\; \mu\cdot T. \;,$$

and

$$\begin{array}{ccc}
\dfrac{\ker\left(\overline{\partial}\colon \wedge^{\bullet,\bullet}\tilde{X}\to\wedge^{\bullet,\bullet+1}\tilde{X}\right)}{\mathrm{im}\left(\overline{\partial}\colon \wedge^{\bullet,\bullet-1}\tilde{X}\to\wedge^{\bullet,\bullet}\tilde{X}\right)} & \xrightarrow[\simeq]{T.} & \dfrac{\ker\left(\overline{\partial}\colon \mathcal{D}^{\bullet,\bullet}\tilde{X}\to\mathcal{D}^{\bullet,\bullet+1}\tilde{X}\right)}{\mathrm{im}\left(\mathrm{d}\colon \mathcal{D}^{\bullet,\bullet-1}\tilde{X}\to\mathcal{D}^{\bullet,\bullet}\tilde{X}\right)} \\
\Big\uparrow{\scriptstyle \epsilon^*_{\overline{\partial}}} & & \Big\downarrow{\scriptstyle \epsilon_*^{\overline{\partial}}} \\
\dfrac{\ker\left(\overline{\partial}\colon \wedge^{\bullet,\bullet}\tilde{Y}\to\wedge^{\bullet,\bullet+1}\tilde{Y}\right)}{\mathrm{im}\left(\overline{\partial}\colon \wedge^{\bullet,\bullet-1}\tilde{Y}\to\wedge^{\bullet,\bullet}\tilde{Y}\right)} & \xrightarrow[T.]{\simeq} & \dfrac{\ker\left(\overline{\partial}\colon \mathcal{D}^{\bullet,\bullet}\tilde{Y}\to\mathcal{D}^{\bullet,\bullet+1}\tilde{Y}\right)}{\mathrm{im}\left(\overline{\partial}\colon \mathcal{D}^{\bullet,\bullet-1}\tilde{Y}\to\mathcal{D}^{\bullet,\bullet}\tilde{Y}\right)}
\end{array} \;,$$

such that

$$\epsilon_*^{\overline{\partial}}\, T.\, \epsilon^*_{\overline{\partial}} \;=\; \mu\cdot T. \;,$$

and

$$\begin{array}{ccc}
\dfrac{\ker\left(\partial\overline{\partial}\colon \wedge^{\bullet,\bullet}\tilde{X}\to\wedge^{\bullet+1,\bullet+1}\tilde{X}\right)}{\mathrm{im}\left(\partial\colon \wedge^{\bullet-1,\bullet}\tilde{X}\to\wedge^{\bullet,\bullet}\tilde{X}\right)+\mathrm{im}\left(\overline{\partial}\colon \wedge^{\bullet,\bullet-1}\tilde{X}\to\wedge^{\bullet,\bullet}\tilde{X}\right)} & \xrightarrow[\simeq]{T.} & \dfrac{\ker\left(\partial\overline{\partial}\colon \mathcal{D}^{\bullet,\bullet}\tilde{X}\to\mathcal{D}^{\bullet+1,\bullet+1}\tilde{X}\right)}{\mathrm{im}\left(\partial\colon \mathcal{D}^{\bullet-1,\bullet}\tilde{X}\to\mathcal{D}^{\bullet,\bullet}\tilde{X}\right)+\mathrm{im}\left(\mathrm{d}\colon \mathcal{D}^{\bullet,\bullet-1}\tilde{X}\to\mathcal{D}^{\bullet,\bullet}\tilde{X}\right)} \\
\Big\uparrow{\scriptstyle \epsilon^*_{BC}} & & \Big\downarrow{\scriptstyle \epsilon_*^{BC}} \\
\dfrac{\ker\left(\partial\overline{\partial}\colon \wedge^{\bullet,\bullet}\tilde{Y}\to\wedge^{\bullet+1,\bullet+1}\tilde{Y}\right)}{\mathrm{im}\left(\partial\colon \wedge^{\bullet-1,\bullet}\tilde{Y}\to\wedge^{\bullet,\bullet}\tilde{Y}\right)+\mathrm{im}\left(\overline{\partial}\colon \wedge^{\bullet,\bullet-1}\tilde{Y}\to\wedge^{\bullet,\bullet}\tilde{Y}\right)} & \xrightarrow[T.]{\simeq} & \dfrac{\ker\left(\partial\overline{\partial}\colon \mathcal{D}^{\bullet,\bullet}\tilde{Y}\to\mathcal{D}^{\bullet+1,\bullet+1}\tilde{Y}\right)}{\mathrm{im}\left(\partial\colon \mathcal{D}^{\bullet-1,\bullet}\tilde{Y}\to\mathcal{D}^{\bullet,\bullet}\tilde{Y}\right)+\mathrm{im}\left(\overline{\partial}\colon \mathcal{D}^{\bullet,\bullet-1}\tilde{Y}\to\mathcal{D}^{\bullet,\bullet}\tilde{Y}\right)}
\end{array} \;,$$

such that

$$\epsilon_*^{BC}\, T.\, \epsilon^*_{BC} \;=\; \mu\cdot T. \;.$$

Since $T.$ are isomorphisms in cohomology, one gets that

$$\epsilon^*_{dR}\colon H^\bullet_{dR}\left(\tilde{X};\mathbb{R}\right)\to H^\bullet_{dR}\left(\tilde{Y};\mathbb{R}\right)\;,\qquad \epsilon^*_{\overline{\partial}}\colon H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{X}\right)\to H^{\bullet,\bullet}_{\overline{\partial}}\left(\tilde{Y}\right)\;,\qquad \text{and}\qquad \epsilon^*_{BC}\colon H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right)\to H^{\bullet,\bullet}_{BC}\left(\tilde{Y}\right)\;.$$

are injective. □

Corollary 3.2. *Let $\tilde{Y}$ and $\tilde{X}$ be compact complex orbifolds of the same dimension, and let $\epsilon\colon \tilde{Y}\to\tilde{X}$ be a proper surjective morphism of complex orbifolds. If $\tilde{Y}$ satisfies the $\partial\overline{\partial}$-Lemma, then also $\tilde{X}$ satisfies the $\partial\overline{\partial}$-Lemma.*

Proof. One has the commutative diagram

$$\begin{array}{ccc}
H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) & \xrightarrow[1:1]{\epsilon^*_{BC}} & H^{\bullet,\bullet}_{BC}\left(\tilde{Y}\right) \\
\Big\downarrow{\scriptstyle \mathrm{id}^*_{\tilde{X}}} & & {\scriptstyle 1:1}\Big\downarrow{\scriptstyle \mathrm{id}^*_{\tilde{Y}}} \\
H^{\bullet}_{dR}\left(\tilde{X};\mathbb{C}\right) & \xrightarrow[1:1]{\epsilon^*_{dR}} & H^{\bullet}_{dR}\left(\tilde{Y};\mathbb{C}\right)
\end{array}$$

where $\mathrm{id}^*_{\tilde{X}}\colon H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right)\to H^{\bullet}_{dR}\left(\tilde{X};\mathbb{C}\right)$ and $\mathrm{id}^*_{\tilde{Y}}\colon H^{\bullet,\bullet}_{BC}\left(\tilde{Y}\right)\to H^{\bullet}_{dR}\left(\tilde{Y};\mathbb{C}\right)$ are the natural maps induced in cohomology by the identity. Since $\mathrm{id}^*_{\tilde{Y}}\colon H^{\bullet,\bullet}_{BC}\left(\tilde{Y}\right)\to H^{\bullet}_{dR}\left(\tilde{Y};\mathbb{C}\right)$ is injective by the assumption that $\tilde{Y}$ satisfies the $\partial\overline{\partial}$-Lemma,

and $\epsilon^*_{BC} \colon H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) \to H^{\bullet,\bullet}_{BC}\left(\tilde{Y}\right)$ and $\epsilon^*_{dR} \colon H^{\bullet}_{dR}\left(\tilde{X};\mathbb{C}\right) \to H^{\bullet}_{dR}\left(\tilde{Y};\mathbb{C}\right)$ are injective by Theorem 3.1, we get that also $\mathrm{id}^*_{\tilde{X}} \colon H^{\bullet,\bullet}_{BC}\left(\tilde{X}\right) \to H^{\bullet}_{dR}\left(\tilde{X};\mathbb{C}\right)$ is injective, and hence $\tilde{X}$ satisfies the $\partial\overline{\partial}$-Lemma. $\square$

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